NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
4 Washington Place, New York 3, N. Y.

AMG-NYU No. 52 Open ~~Copy No. 2~~
C.1

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
4 Washington Place, New York 3, N.Y.

NOTE ON A HEAT TRANSFER PROBLEM

A solution $u = u(r,t)$ of the differential equation

$$1) \quad \frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right)$$

is to be found for $r \geq r_o$, $t \geq 0$, satisfying the initial condition

$$u = 0 \text{ for } t = 0,\ r > r_o$$

and the boundary condition

$$u = u_o \text{ for } r = r_o,\quad t \geq 0.$$

Our main interest is to find out at what time, t, at a given place, r, a certain percentage of the final temperature, u_o, is reached. Also some information about the quantity

$$\left.\frac{\partial u}{\partial r}\right|_{r = r_o}$$

proportional to the heat flux is desired.

To replace equation (1) by a difference equation, we proceed as follows:

Let $u_{n,m} = u(r_n,t_m)$ and replace 1) by

$$2) \quad \frac{u_{n,m+1} - u_{n,m}}{\Delta t_m} = \alpha\left[\frac{\dfrac{u_{n+1,m} - u_{n,m}}{\Delta r_n} - \dfrac{u_{n,m} - u_{n-1,m}}{\Delta r_{n-1}}}{\frac{1}{2}(\Delta r_n + \Delta r_{n-1})}\right] + \frac{\Delta r_n}{\Delta r_n} \cdot \frac{\alpha}{r_n} \cdot \frac{u_{n+1,m} - u_{n-1,m}}{\Delta r_n + \Delta r_{n-1}},$$

Where

$$\Delta r_n = r_{n+1} - r_n\ ;\quad \Delta t_m = t_{m+1} - t_m\ .$$

Now, using the following relationship

3) $$\Delta r_n = \frac{r_n}{\beta} \quad , \Delta r_{n-1} = \frac{r_{n-1}}{\beta}$$

$$r_n = r_{n-1}(1 + \frac{1}{\beta}), \therefore \; r_{n-1} = \frac{\beta}{1 + \beta} r_n$$

and from 3)

4) $$\Delta r_{n-1} = \frac{r_{n-1}}{\beta} = \frac{1}{\cancel{\beta}} \; \frac{\cancel{\beta}}{1 + \beta} r_n = \frac{\beta}{1 + \beta} \Delta r_n$$

Therefore,

5) $$\Delta r_n + \Delta r_{n-1} = \frac{1 + 2\beta}{1 + \beta} \Delta r_n$$

We now substitute 4) and 5) in 2) and obtain

6) $$\frac{u_{n,m+1} - u_{n,m}}{\alpha \Delta t_m} = \frac{u_{n+1,m} - u_{n,m}}{\Delta r_n^2 \, (\frac{2\beta + 1}{2\beta + 2})} - \frac{u_{n,m} - u_{n-1,m}}{\Delta r_n^2 \; \frac{\beta(2\beta + 1)}{2(\beta + 1)^2}}$$

$$+ \frac{\Delta r_n}{r_n \, \Delta r_n} \quad \frac{u_{n+1,m} - u_{n-1,m}}{(\frac{2\beta + 1}{\beta + 1}) \Delta r_n}$$

Letting

7) $$\frac{\alpha \Delta t_m}{(\Delta r_n)^2} = \gamma$$, we substitute in 6) and obtain

8) $$u_{n,m+1} = u_{n,m} + \frac{\gamma(2\beta + 2)}{(2\beta + 1)} (u_{n+1,m} - u_{n,m})$$

$$- \frac{2\gamma(\beta + 1)^2}{(2\beta + 1)} (u_{n,m} - u_{n-1,m})$$

$$+ \frac{\gamma(\beta + 1)}{\beta(2\beta + 1)} \quad (u_{n+1,m} - u_{n-1,m})$$

And collecting terms:

$$\boxed{\begin{array}{l} u_{n,m+1} = C\,u_{n-1,m} + (1-2C)\,u_{n,m} + C\,u_{n+1,m} \\ \text{where} \qquad C = \dfrac{\gamma(\beta+1)}{\beta} \end{array}} \tag{9}$$

The results obtained by setting $C = \frac{1}{2}$ are not as reliable as those with smaller C:

$\gamma = \frac{1}{4}$, $\beta = 5$: coefficient $C = .3$, and for a finer approximation

$\gamma = \frac{1}{4}$, $\beta = 10$: coefficient $C = .275$.

Graphs were drawn for some of the surfaces at a constant distance from the center of the hole. They indicate that for a fixed time the temperature decreases exponentially as we move away from the center of the hole and that for a fixed surface the temperature increases very slowly to the temperature maintained at hole.

The small difference observed when comparing the results of both calculations, indicates that the interval for $\beta = 10$ can be safely employed, if we do not venture too far.

A rough estimation of the number of values necessary to indicate the state of the surface whose distance is 20 times the hole radius is

500 values or 1500 multiplications for $\beta = 5$

1000 values or 3000 multiplications for $\beta = 10$.

We, therefore, recommend that serious consideration be

given to the evaluation of the following formula[*] which represents the exact solution:

$$10)\quad u = u_0 + \frac{2u_0}{\pi}\int_0^{\infty} e^{-\alpha u^2 t}\,\frac{J_0(ru)\,Y_0(r_0u) - J_0(r_0u)Y_0(ru)}{J_0^2\,(r_0u) + Y_0^2\,(r_0u)}\cdot\frac{du}{u}$$

where J_0 is the Bessel function of order zero

Y_0 is the Neumann function of order zero

r_0 is the radius of the hole

u_0 is the temperature of hole

The numerical integration of the integral appearing in (10) is straightforward, if one makes use of the following approximate formula, valid for small ε[**]:

$$11)\quad \int_0^{\varepsilon}[\text{integrand in (10)}]du \backsim \log\frac{r_0}{r}\,\left[\arctan\left(a\log\frac{r_0}{r} + C\right)\right]_0^{\varepsilon}$$

where $a = \frac{2}{\pi} = .6366$, $C = \frac{2}{\pi}(\gamma - \log 2) = -.07380$.

We used (11) in evaluating (10) for the particular case $r = 2r_0$, $\alpha t = 4r^2$ and obtained $u = .66u_0$.

---

[*] H.S. Carslaw and J.C. Jaeger - Some Two-Dimensional Problems in Conduction of Heat with Circular Symmetry - Proceedings of London Mathematical Society - Series 2, volume 46, 1940, pp. 361-388, in particular, pp. 367-368.

[**] For $0 < \varepsilon \leq u$, we may use Simpson's rule.

CASE I

$$\frac{\alpha \Delta t_m}{(\Delta r_n)^2} = \frac{1}{4} \quad , \quad \frac{\Delta r_n}{r_n} = \frac{1}{5}$$

$$C = .3$$

t\r	r_0	r_1	r_2	r_3	r_4	r_5	..	..	..	.		..
t_0	100	0										
t_1	100	30.0	0									
t_2	100	42.0	9.0	0								
t_3	100	49.5	16.2	2.7	0							
..	100	54.7	22.1	5.94	.81	0						
..	100	58.5	27.0	9.25	2.11	.24	0					
..	100	61.5	31.1	12.4	3.69	.73	.07	0				
..	100	63.9	34.6	15.4	5.42	1.42	.25	.02	0			
..	100	65.9	37.6	18.2	7.21	2.27	.53	.08	.01	0		
..	100	67.6	40.3	20.7	9.03	3.23	.92	.19	.03	0	0	
..	100	69.1	42.6	23.1	10.8	4.28	1.39	.36	.07	.01	0	0
..	100	70.4	44.7	25.3	12.5	5.37	1.95	.58	.14	.03	0	0

CASE II

$$\frac{\alpha \Delta t_m}{(\Delta r_n)^2} = \frac{1}{4} \quad , \quad \frac{\Delta r_n}{r_n} = \frac{1}{10}$$

$$C = .275$$

t \ r	r_o	r_1	r_2	r_3	r_4	r_5	..	..	..	..	..	..
t_o	100	0										
t_1	100	27.5	0									
t_2	100	39.9	7.56	0								
t_3	100	47.5	14.4	2.08	0							
t_4	100	52.8	20.1	4.90	.57	0						
..	100	56.8	24.9	7.89	1.60	.16	0					
..	100	59.9	29.0	10.8	2.93	.51	.04	0				
..	100	62.4	32.5	13.6	4.42	1.05	.16	.01	0			
..	100	64.5	35.5	16.3	6.02	1.73	.36	.05	0	0		
..	100	66.3	38.2	18.8	7.67	2.53	.65	.12	.01	0	0	
..	100	67.8	40.6	21.1	9.32	3.43	1.02	.24	.04	0	0	0
..	100	69.2	42.7	23.2	10.9	4.39	1.47	.40	.08	.01	0	0

DATE DUE

~~JUL 2 7 1952~~			
GAYLORD			PRINTED IN U.S.A.

Send your
Represen
Bet
YAWMAN AND ERBE MFG. CO.
Main Factories and Executive Offices
ROCHESTER, N. Y.
Branches and Agents in all Principal Cities

NYU c.1
AMG-52
Isaacson

Note on a heat transfer
problem.

NYU c.1
AMG-52
Isaacson

AUTHOR
Note on a heat transfer

TITLE
problem

DATE DUE	BORROWER'S NAME
~~JUL 27~~	Ahmed

N. Y. U. Institute of
Mathematical Sciences
~~25 Waverly Place~~
New York 3, N. Y.
4 Washington Place

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
4 Washington Place, New York 3, N. Y.